Table 1 Average sizes of collected particles

			without NaN_3	with NaN_3
Arithmetic mean, on number basis	d_n	$= (\Sigma fd)$	$= 0.34\ \mu$	$0.27\ \mu$
Surface area, on number basis	d_s	$= (\Sigma fd^2)^{1/2}$	$= 0.67\ \mu$	$0.39\ \mu$
Volume, on number basis	d_v	$= (\Sigma fd^3)^{1/3}$	$= 1.49\ \mu$	$0.87\ \mu$
Geometric mean, on weight basis	d_m	$= \Sigma fd^4/\Sigma fd^3$	$= 13.70\ \mu$	$12.81\ \mu$

The mass deposited per unit area

$$m_D = \frac{\pi\rho}{6}\sum_{\Delta d} d^3 \frac{\Delta n}{\Delta d}$$

where the bulk density is $\rho = 4.6$ g cm^{-3}, agreed within 20% with that value calculated from the total ejected mass M and the dilution factor for $\cos^2$ distribution[7] of mass-flow densities over flow angles θ at the nozzle distance x on the centerline;

$$m_D = \left[1\Big/\int_0^{\theta_m} \cos^2\frac{\pi}{2}\frac{\theta}{\theta_m}\sin\theta d\theta\right]\cdot M/2\pi x^2 = 1.02M/x^2$$

This consistency confirms the applicability of the flow picture and that representative samples were obtained also of the larger particles, which are seldom numberwise but still determine the mass weighted average.

With a few exceptions the droplets seemed to be a homogeneous mixture of all three reaction products, as they had uniformly the same reddish colour[8] (determined by Cu) and electron diffractograms did often not reveal any crystal structure,[10] which also is absent in many salt aerosols.[12]

Results and Discussion

By inspection of the two micrographs (Fig. 3) one recognizes that addition of controlled amounts of N_2 produces much finer particles than observed for reaction mixtures, where the N_2 contents depend on accidental impurities (approximately 500 ppm N_2 in commercial Ba and decomposition products from ablative insulation of the reaction vessel). This is quantitatively born out by the two size-frequency curves (Fig. 2), each being constructed with data of many effusion experiments. The various physically relevant average sizes[11] are, with the normalized distribution function

$$f = (\Delta n/\Delta d)/(\Sigma\Delta n/\Delta d)$$

given in Table 1.

These size distributions from this expansion into vacuum indicate a larger abundance of smaller particles than observed in rocket exhaust plumes,[13] in which case small particles are supposed to coalesce to larger ones during the dwell time upstream of the nozzle.[14] By the same token, one may surmise that particle fragmentation of these jets with an initial weight fraction of condensed matter of more than 95% takes place mainly downstream of the nozzle where agglomeration is no more possible.

To be sure, a detailed analysis[7] of the nozzle flow and of the pressure change inside an accelerated reaction vessel has shown, that the liquid is dispersed already before passage through the nozzle, but with much larger mean droplet size than derived from the terminal size distribution reported here.

The fragmentation mechanism is normally sought in shear and turbulence, effects which might only be of minor importance here because of the low Reynolds numbers.[5] The enhancement of atomization upon addition of NaN_3 rather suggests, that N_2 is partially solved by the liquid metal at the high pressure inside the reaction vessel and assists in further droplet break-up upon discharge. The vapor pressure of Ba is probably too low under these conditions to exceed the surface pressure $\Pi \gtrsim 10^6$ dyne/cm^2 of particles with $d \simeq 10\ \mu$. Even though this analysis of droplet sizes has been made without time resolution and only on the centerline of the jet axis (with increasing flow angle a shift of the size distribution toward smaller diameters is expected), it indicates how to control the size distribution by simple means. The large concomitant increase in vapor yield of Ba has been demonstrated elsewhere.[7]

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A Transformation between Axisymmetric and Two-Dimensional Turbulent Boundary Layers

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IN Ref. 1, Mangler gives a method for relating the properties of a laminar boundary layer on a body of revolution to those of a corresponding two-dimensional flow. There did

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not seem to be a method for turbulent flow. After this paper was first submitted for publication, however, attention was called to Ref. 2, which does not seem to be generally known, in which Mangler presents one for turbulent incompressible flow. In the present Note, a transformation for turbulent compressible flow is derived in a somewhat different way and a few applications are given. In addition, it is shown that when a power type friction formula holds, the equations of motion and enthalpy can be nondimensionalized in such a way that useful conclusions can be drawn concerning the effect of Reynolds number on friction, heat transfer, recovery factor, separation point, and velocity and total enthalpy profiles.

To derive a transformation for turbulent compressible flow Mangler's transformation for laminar flow is generalized by writing the mean-turbulent quantities $\bar{\rho}$, $\bar{u}$, $\bar{\rho}\bar{v} + \overline{\rho' v'}$, etc., where ρ, u, v, etc., appear in laminar flow. (For a complete derivation see Ref. 3.) Moreover, Mangler's transformation between x_1, in two-dimensional flow and x on a body of revolution is generalized to

$$x_1 = \int_0^x f\left(\frac{r_w}{L}\right) dx \tag{1}$$

where $r_w(x)$ is the radius of the body of revolution and L is a fixed reference length. By use of the transformation the equation of motion for a thin boundary layer on a body of revolution is transformed to

$$f\left(\frac{r_w}{L}\right)\left[\bar{\rho}_1\bar{u}_1\frac{\partial \bar{u}_1}{\partial x_1} + (\bar{\rho}_1\bar{v}_1 + \overline{\rho'_1 v'_1})\frac{\partial \bar{u}_1}{\partial y_1}\right] = -\frac{d\bar{p}_1}{dx_1} f\left(\frac{r_w}{L}\right) + \frac{r_w}{L}\frac{\partial \tau}{\partial y_1} \tag{2}$$

where $(\ )_1$ denotes two-dimensional flow. To remove the (r_w/L) terms and thus make Eq. (2) the boundary-layer equation for two-dimensional flow it is sufficient that

$$\tau = \tau_1(r_w/L)^{-1}f(r_w/L) \tag{3}$$

To find $f(fr_w/L)$ write Eq. (3) for $y = 0$ and use the transformation relations $\rho_e = \rho_{e1}$ and $u_e = u_{e1}$, where $(\ )_e$ denotes "at the outer edge of the boundary layer." The result is

$$f(r_w/L) = (r_w/L)(C_f/C_{f1}) \tag{4}$$

where C_f and C_{f1} are the friction coefficients at corresponding x and x_1. To obtain a simple explicit relation for $f(r_w/L)$, the friction coefficient is expressed as a function of the local Mach number M_e, local wall to stream temperature ratio T_w/T_e, and the shape of the nondimensional velocity profile, multiplied by a function of the local boundary-layer momentum thickness Reynolds number Re_θ; thus for axisymmetric flow

$$C_f = k(M_e,(T_w/T_e),H_u)Re_\theta{}^{-n} \tag{5}$$

and for two-dimensional flow,

$$C_{f1} = k(M_e,(T_w/T_e),H_u)Re_{\theta 1}{}^{-n} \tag{6}$$

By writing explicit expressions for C_f and C_{f1} the physics of the two flows are specified and the transformation determined. The use of the same form for C_f and C_{f1} in Eqs. (5) and (6) implies that the ratio of the boundary-layer thickness to r_w is small enough for any effect of transverse curvature that is independent of Re_θ, M_e, T_w/T_e, and the nondimensional profile (denoted by H_u) is negligible. The symbol H_u in Eqs. (5) and (6) denotes the shape of the nondimensional velocity profile; at this point, it is not required that the velocity profiles be a single parameter family with H_u as the parameter.

At corresponding x and x_1, the transformation between axisymmetric and two-dimensional flow makes $M_e = M_{e1}$, $T_e = T_{e1}$, $T_w = T_{w1}$, and also makes the nondimensional profile at x_1 in two-dimensional flow the same as that at x in axisymmetric flow. Therefore, k is the same at corresponding x and x_1. Then Eqs. (5) and (6) hold for nonzero pressure gradient and for compressible flow. For example, if the reference enthalpy method for compressible flow (Ref. 4, p. 670) is combined with the Ludwig-Tillman formula for pressure gradient flow (Ref. 4, p. 635) the result is (Ref. 5)

$$(C_f/2) = (0.123/Re_\theta{}^{0.268})10^{-0.678H_u}(\mu_r/\mu_e)^{0.268}(\rho_r/\rho_e) \tag{7}$$

where

$$k = 0.123 \cdot 10^{-0.678H_u}\,(\mu_r/\mu_e)^{0.678}(\rho_r/\rho_e) \text{ and } n = 0.268$$

The symbol H_u in Eq. (7) is now the shape factor for turbulent velocity profiles and specifies the nondimensional profile.

From Eqs. (4–6) it follows that

$$f(r_w/L) = (r_w/L)(\theta_1/\theta)^n \tag{8}$$

But from the definition of the momentum thicknesses θ_1 and θ and the transformation between $(\ )$ and $(\ )_1$, it follows that $\theta_1/\theta = r_w/L$. Then Eq. (8) becomes

$$f(r_w/L) = (r_w/L)^{n+1} \tag{9}$$

For laminar flow $n = 1$, and Mangler's well known transformation results. For laminar flow the friction formulas Eqs. (5) and (6) with $n = 1$ are exact. For turbulent flow, however, Eqs. (5) and (6) are approximate and therefore so is the transformation Eq. (9). Although n is assumed to be constant over some Reynolds number range, n actually decreases slowly with increase in Reynolds number and therefore with increase in the "turbulence" of the flow. Thus, n is unity for laminar flow, about $\frac{1}{4}$ for low Reynolds number turbulent flows, and becomes zero as $Re_\theta \to \infty$ or the flow becomes completely rough.

The equations of motion, total enthalpy, and continuity can be written in a nondimensional form so that the Reynolds number does not appear, just as for laminar flow (Ref. 4, p. 136). Thus, let

$$\begin{gathered} x_* = x/L,\ v_* = (\bar{v}/u_\infty)Re_L{}^q,\ p_* = (p - p_\infty)/\rho_\infty u_\infty{}^2, \\ r_{w*} = r_w/L,\ (\overline{\rho' v'})_* = (\overline{\rho' v'}/\rho_\infty u_\infty)Re_L{}^q, \\ \tau_* = (\tau/\rho_\infty u_\infty{}^2)Re_L{}^s,\ y_* = (y/L)Re_L{}^q, \\ \rho_* = \bar{\rho}/\rho_\infty,\ I_* = I/I_\infty, \\ u_* = \bar{u}/u_\infty,\ \mu_* = \bar{\mu}/\mu_\infty,\ Q_* = (Q/\rho_\infty u_\infty I_\infty)Re_L{}^s \end{gathered} \tag{10}$$

Then the equation of motion becomes, with $[\bar{\mu}(\partial\bar{u}/\partial y) - \bar{\rho}\overline{u'v'}] = \tau$

$$\rho_* u_* \frac{\partial u_*}{\partial x^*} + [\rho_* v_* + (\overline{\rho' v'})_*]\frac{\partial u_*}{\partial y_*} = -\frac{dp_*}{dx_*} + \frac{\partial \tau_*}{\partial y_*} \tag{11}$$

where it was necessary to take $s = q$ in Eq. (10) to eliminate Re_L in Eq. (11). Now write Eq. (5) so that τ_{*w} appears. The result is

$$\tau_{*w} = (\rho/\rho_\infty)(u_e/u_\infty)^2 k(u_{*e}\rho_{*e}\theta_*/\mu_{*e})^{-n}Re_L{}^{q(1+n)-n}$$

Thus, τ_{w*} is independent of Re_L if $q = n/n + 1$. The equation for the total enthalpy I with $Q = \bar{k}_c\partial\bar{T}/\partial y - \bar{\rho}\overline{v'h'}$, becomes

$$\rho_* u_* \frac{\partial I_*}{\partial x_*} + [\rho_* v_* + (\overline{\rho' v'})_*]\frac{\partial I_*}{\partial y} = \frac{\partial Q_*}{\partial y_*} + \frac{u_\infty{}^2}{I_\infty}\frac{\partial u_*\tau_*}{\partial y_*} \tag{12}$$

The continuity equation takes a corresponding form.

The boundary conditions become $u_{*e}(x_*)$, $I_{*e}(x^*)$, $p_{*e}(x_*)$ at the outer edge of the boundary layer and $u_* = 0$, $v_* = v_*(x_*)$, and $I_{*w} = I_{*w}(x_*)$ or $Q_{*w} = Q_{*w}(x_*)$ at the wall. Because Eqs. (11), (12), and the continuity equation are independent of Reynolds number it follows that both the nondimensional velocity and total enthalpy profiles are fixed functions of $y/L\ Re_L{}^{n/n+1}$ at a fixed x/L when the boundary conditions are independent of Reynolds number and u_∞^2/I_∞ is fixed. Also, $\theta/L\ Re_L^{n/n+1}$, $\delta^*/L\ Re_L^{n/n+1}$ and the other nondimensional thicknesses are independent of Re_L. Moreover, the local shear stress and local heat transfer vary as $Re_L^{-n/n+1}$; therefore the total friction stress and the total heat transfer vary as $Re_L^{-n/n+1}$ if the boundary layer is either entirely laminar or entirely turbulent.

Because τ_* is independent of Reynolds number, the skin-friction drops to zero at a value of x/L that is independent of Reynolds number. Therefore, the separation point also is independent of Reynolds number. The same conclusion follows from the result that the nondimensional velocity profiles are independent of Reynolds number. For these results to be correct, the friction coefficient must be expressible as a power function of Re_θ; this is exact for laminar flow but only approximate for turbulent flow. Moreover, the nondimensional velocity profiles and the nondimensional thicknesses at the initial point of the boundary layer must be independent of Reynolds number.

If $Q_{*w} = 0$, u_∞^2/I_∞ fixed, and the other boundary conditions made independent of Re_L a solution of the nondimensional equations gives $I_{*a}(x_*)$, the value for zero heat transfer. Under these conditions, $I_{*a}(x_*)$ is independent of Re_L. From the definition of the recovery factor r, it follows that

$$I_{*a}(x_*) = I_{*e}(x_*) - (u_{*e}^2/2)(1 - r)(u_\infty^2/I_\infty)$$

Therefore when u_∞^2/I_∞ is fixed and the boundary conditions are independent of Re_L the recovery factor is independent of Re_L.

By using Eqs. (5), (6), and (9) with $\rho_e = \rho_{e1}$, $u_e = u_{e1}$, there is obtained

$$C_{f1}Re_{x1}^{n/(n+1)} = C_fRe_x^{n/(n+1)}\left[\int_0^{x/L}\left(\frac{r_w}{L}\right)^{n+1} d\frac{x}{L}\bigg/\frac{x}{L}\left(\frac{r_w}{L}\right)^{n+1}\right]^{n/(n+1)} \quad (13)$$

at corresponding x and x_1. A relation for Stanton numbers is obtained from Eq. (13) by replacing C_f by St and C_{f1} by St_1. Relations between various boundary-layer thicknesses in axisymmetric and two-dimensional flows are derived in the same way.

For a cone, $r_w/L = ax/L$ and ρ_e and u_e are constant. Because $\rho_e = \rho_{e1}$ and $u_e = u_{e1}$, ρ_{e1} and u_{e1} also are constant. Therefore the corresponding two-dimensional flow is that over a flat plate at zero angle of attack. Equation (13) then becomes

$$C_{f1}Re_{x1}^{n/(n+1)} = C_fRe_x^{n/(n+1)}(n + 2)^{-n/(n+1)} \quad (14)$$

at corresponding x and x_1 with $M_e = M_{e1}$, $T_e = T_{e1}$, $T_w = T_{w1}$, $p_e = p_{e1}$.

For a plate or cone with constant surface temperature the reference length L can be replaced by x. Therefore, the conclusion that the local shear stress varies as $Re_L^{-n/n+1}$ can be expressed as

$$C_{f1}Re_{x1}^{n/(n+1)} = C_1 \text{ and } C_fRe_x^{n/(n+1)} = C_2$$

Consequently, $C_{f1}Re_{x1}^{n/(n+1)}$ is independent of x_1 and $C_fRe_x^{n/(n+1)}$ is independent of x. Therefore Eq. (14) holds for all combinations of x and x_1. In particular for $Re_f = Re_{x1}$

$$C_f/C_{f1} = (n + 2)^{n/(n+1)} \quad (15)$$

and for

$$C_f = C_{f1},$$
$$Re_{x1}/Re_x = 1/(n + 2) \quad (16)$$

These relations are usually obtained by using the boundary-layer momentum equation for a cone and that for a plate, together with a power law friction formula, and the assumption of similar velocity profiles on cone and plate.

Van Driest[6] obtains the relation

$$Re_{x1}/Re_x = \tfrac{1}{2} \quad (17)$$

for turbulent flow instead of Eq. (16). In going from Eq. (16) to Eq. (17) of Ref. 6 and from Eq. (19) to Eq. (20) of Ref. 6, it was assumed that a quantity "a" was very large. This means that C_f was very small and thus that Re_θ was very large. Because the rate of decrease of C_f with increase in Reynolds number decreases as the Reynolds number increases, $n \to 0$ as $Re_\theta \to \infty$. Then Eq. (16) approaches Eq. (17). Therefore Eq. (17) is a limiting relation for very large Re_θ.

By taking the boundary conditions at infinity for the cone boundary layer to be those at the outer edge of the boundary layer and equal to those at the outer edge of the flat plate boundary layer, it can be shown that the recovery factor on a cone equals that on a flat plate and that both are independent of Reynolds number.

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Some Effects of Acceleration on the Turbulent Boundary Layer

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IMPLICIT in most boundary-layer analyses is the assumption that the boundary layer entrains fluid from the freestream, i.e., that the proper outer edge condition for the boundary layer is the local freestream. That this is not necessarily true has been demonstrated in the case of supersonic flow over blunted bodies where the boundary-layer development takes place within the shock layer formed by a leading edge bow shock.[1] An attempt will be made in the present Note to demonstrate experimentally that boundary-layer entrainment of the local freestream also may not be a valid assumption for the case of strongly accelerated flows in general. In this instance, the boundary-layer thinning can become so severe that the boundary layer may be considered to develop within the wake of the upstream boundary layer rather than in the freestream.

Experimental Procedure

Boundary-layer profiles and wall-static pressures were measured in the two-dimensional convergent-divergent test

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